Hauer's Tropes and the Enumeration of Twelve-Tone Hexachords

Polytrope

2008 v 3

The 'tropes' (hexachordal partitions) of Josef Matthias Hauer[1] (1883-1959) afford a compact enumeration of the 6-chord types in a 12-pitch-class context.[2] Since 36 of the 44 tropes[3] comprise two hexachords of different types, a listing of the 44 tropes shows clearly all 80 hexachord types.

Furthermore, if (as in Figures 2-8 here) the tropes are arranged in order of George Perle's enumeration of the hexachords,[4] then hexachord types of the same combinatorial character[5] appear together. This is true because, 1) being complementary, the two halves of a given trope have necessarily the same combinatorial character, and 2) combinatorial character corresponds exactly with the kinds of symmetry which guide Perle's enumeration.

Each component diagram in Figures 2-8 here shows the two complementary hexachords of the Hauer trope whose number appears in the centre of the diagram. The pitch classes belonging to each half-trope are indicated by filled circles or open circles, the position of each circle indicating a corresponding pitch-class number as in Figure 1. If the Hauer number is shown with an overline, then the filled circles represent the second half-trope; otherwise the filled circles represent the first half-trope.

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[1] See `www.musiker.at/sengstschmidjohann/stichwort-trope.php3` . Thanks to Christopher Butterfield for calling my attention to the affinity of my PSTP technique to Hauer's. (See my "Pitch-Symmetric Tetrachordal Partitions" on this website.)

[2] See my "Pitch-Symmetric Tetrachords, etc" on this website, pp 9ff.

[3] All but tropes 1, 8, 17, 19, 24, 34, 41, and 44.

[4] *Serial Composition and Atonality,* University of California Press, (6th ed., rev.) 1991, pp 153-155 ("Six-Note Collections").

[5] I.e. the property of comprising or not the pitch class sets of the first or last half of a tone row which is combinatorial by a particular transformation.

Below each diagram are shown the interval vector[6] common to the two half-tropes and the numbers assigned them in Perle's enumeration (P) and in that of Allen Forte[7] (F). Figures 2 and 3 also show the applicable symbols from my "Pitch-Symmetric Tetrachords, etc".[8]

The choice of transformation for each trope makes the pitch-class numbers agree with those in Perle's enumeration, except for the 'Ai'/'Bi' diagrams in Figures 7 and 8, where my choice emphasizes the relation of pitch-inversion (mirror symmetry in the diagrams) between those and their 'Ap'/'Bp' counterparts. Note that the four hexachords sharing each Perle number here also share an inverval vector. The affixed '=' signs call attention to this. Likewise in Figure 6.

For Figures 2-5, due to the choice of transformation for each trope, either the $11.5 \leftrightarrow 5.5$ diameter or (for P1, P3, P7, P10, P12, P13) the $0 \leftrightarrow 6$ diameter of each diagram is an axis of symmetry (Figures 2 and 3) or antisymmetry (Figures 4 and 5).

To summarize, Figures 2-8 constitute an illustrated listing of the hexachord types by Perle number, showing also, *inter alia,* their Forte numbers and their position among Hauer's tropes.

The appendices complete the cross-indexing of these three enumerations. Appendix A shows the Hauer tropes in their original order along with the Perle and Forte numbers for their constituent hexachords. Appendix B lists the hexachords by Forte number along with their 'prime forms', interval vectors, combinatorial character, Perle numbers, and position among Hauer's tropes.

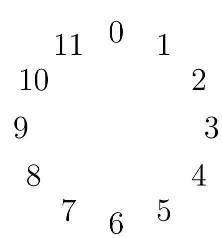


Figure 1: Circle-of-semitones order for Figures.

[6] I.e. the numbers of intervals within the hexachord comprising respectively 1, 2, 3, 4, 5, and 6 semitones.

[7] *The Structure of Atonal Music,* Yale University Press, 1973 (prefixes "6–" omitted). Note that Forte's enumeration does not distinguish between a set and its pitch-inverse; accordingly many Forte numbers apply to more than one half-trope.

[8] On this website.

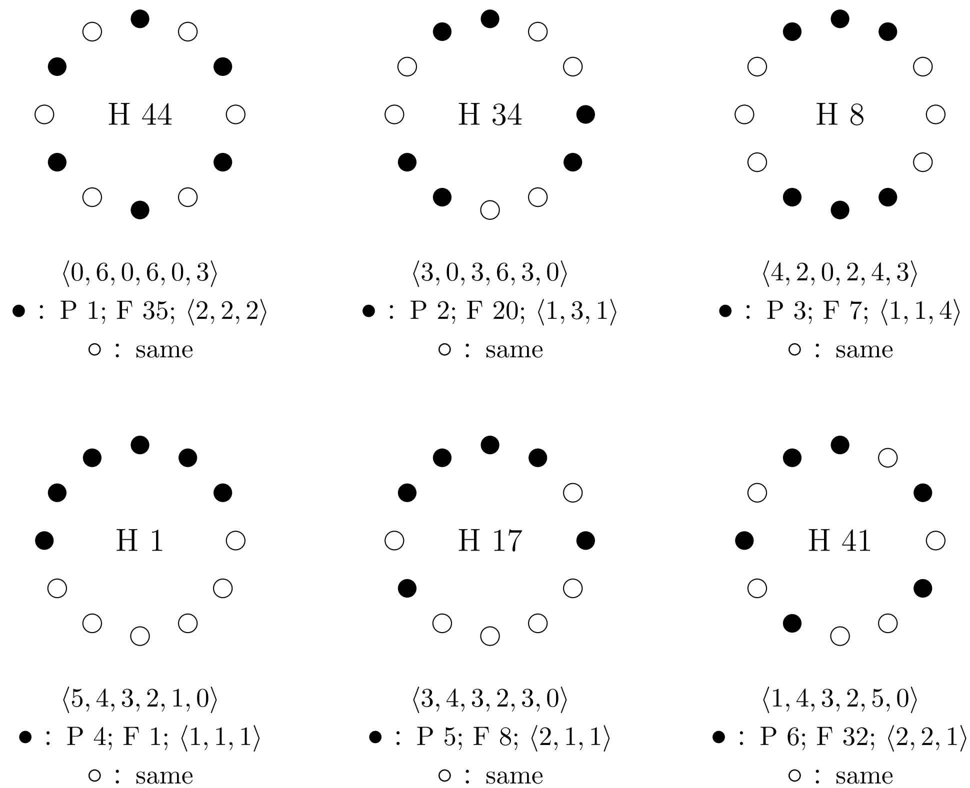


Figure 2: Tropes for Fully Combinatorial Rows

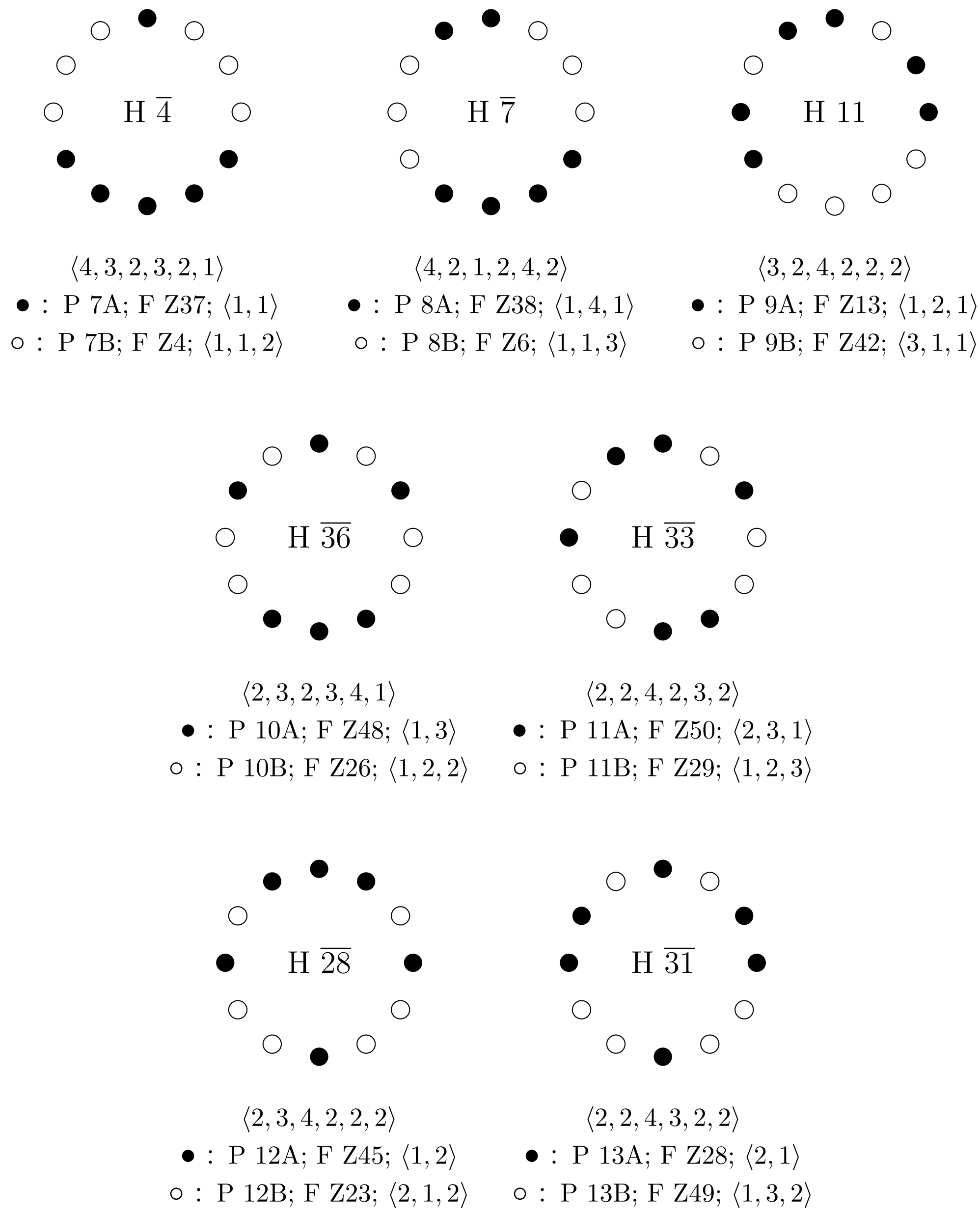


Figure 3: Tropes for Rows Semicombinatorial by Retrograde Inversion.

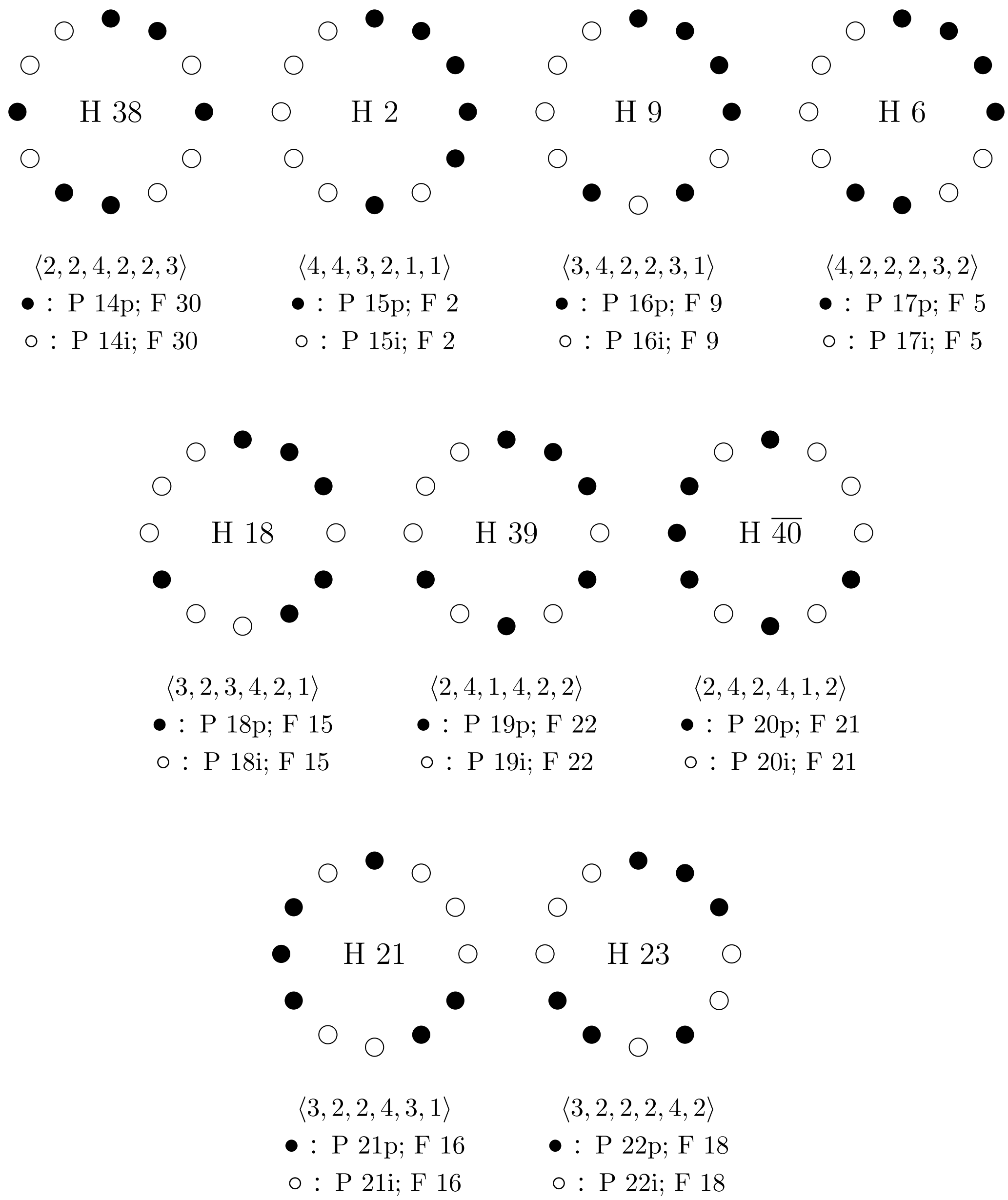
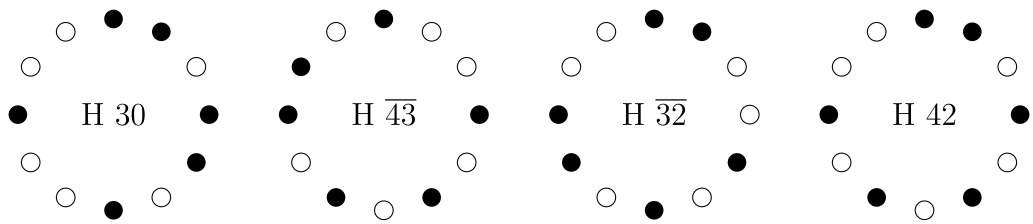
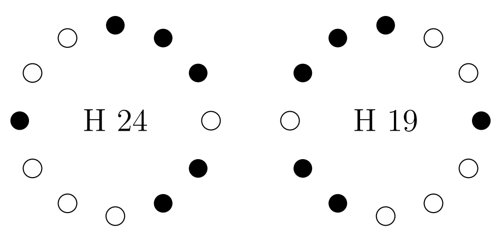


Figure 4: Tropes for Rows Semicombinatorial by Inversion (1 of 2).



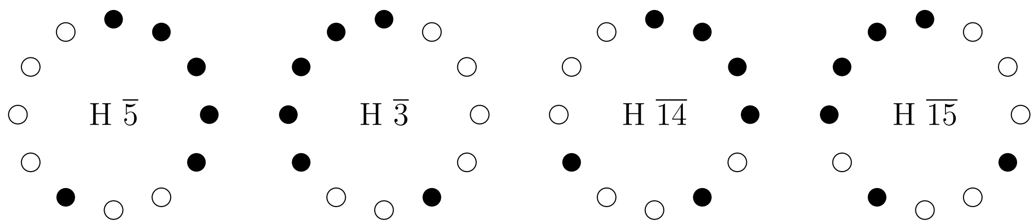
$\langle 2,2,5,2,2,2\rangle$	$\langle 1,4,3,2,4,1\rangle$	$\langle 2,2,3,4,3,1\rangle$	$\langle 1,4,2,4,2,2\rangle$
• : P 23p; F 27	• : P 24p; F 33	• : P 25p; F 31	• : P 26p; F 34
○ : P 23i; F 27	○ : P 24i; F 33	○ : P 25i; F 31	○ : P 26i; F 34

Figure 5: Tropes for Rows Semicombinatorial by Inversion (2 of 2).

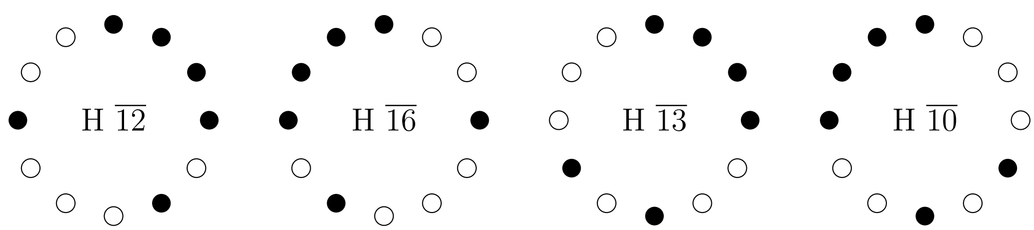


$\langle 3,2,3,4,3,0\rangle =$	$= \langle 3,2,3,4,3,0\rangle$
• : P 27p; F 14	• : P 27i; F 14
○ : same	○ : same

Figure 6: Tropes for Rows Semicombinatorial by Transposition.



$\langle 4,3,3,2,2,1\rangle =$	$= \langle 4,3,3,2,2,1\rangle$	$\langle 3,3,3,2,3,1\rangle =$	$= \langle 3,3,3,2,3,1\rangle$
• : P 28Ap; F Z36	• : P 28Ai; F Z36	• : P 29Ap; F Z40	• : P 29Ai; F Z40
○ : P 28Bp; F Z3	○ : P 28Bi; F Z3	○ : P 29Bp; F Z11	○ : P 29Bi; F Z11



$\langle 3,3,3,3,2,1\rangle =$	$= \langle 3,3,3,3,2,1\rangle$	$\langle 3,3,2,2,3,2\rangle =$	$= \langle 3,3,2,2,3,2\rangle$
• : P 30Ap; F Z39	• : P 30Ai; F Z39	• : P 31Ap; F Z41	• : P 31Ai; F Z41
○ : P 30Bp; F Z10	○ : P 30Bi; F Z10	○ : P 31Bp; F Z12	○ : P 31Bi; F Z12

Figure 7: Tropes for Noncombinatorial Rows (1 of 2).

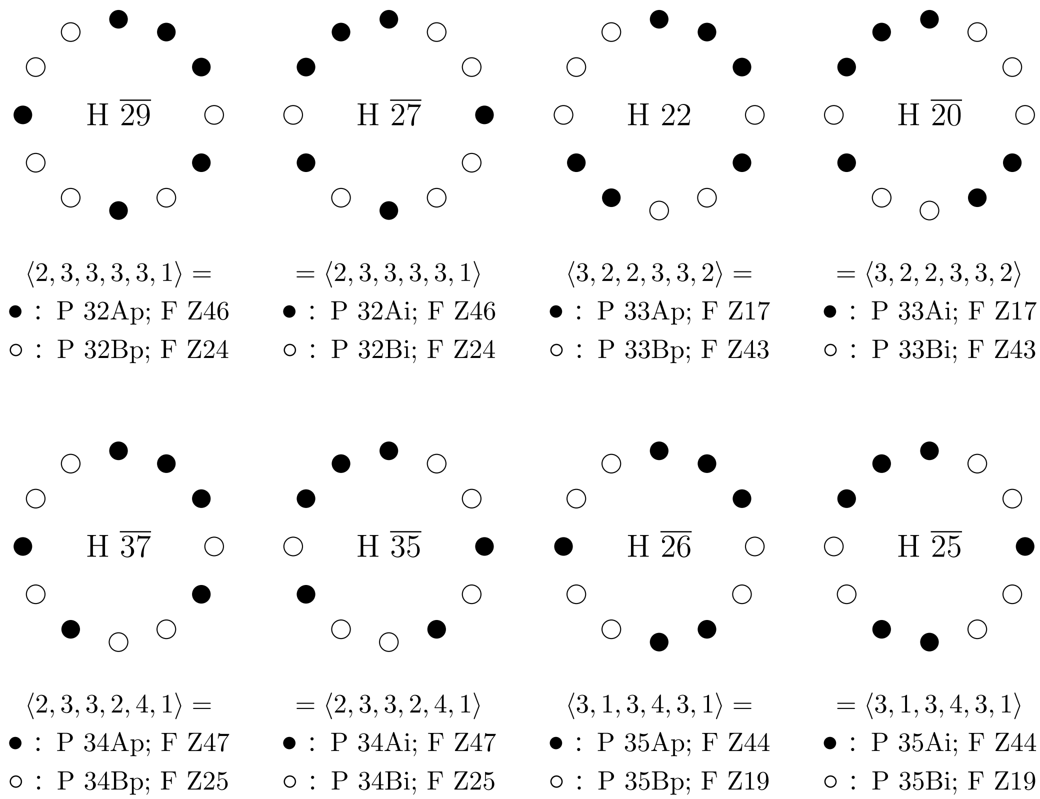


Figure 8: Tropes for Noncombinatorial Rows (2 of 2).

A Hauer's Tropes in Original Order

The diagrams in this appendix (Figures 9-12) are generally as above, but arranged so as to represent directly "TAFEL I" on

`www.musiker.at/sengstschmidjohann/stichwort-trope.php3` .

In particular, assuming pitch-class numbers as in Figure 1, 0 here corresponds to the pitch class D♯=E♭, and closed circles indicate first half-tropes, open circles second half-tropes.

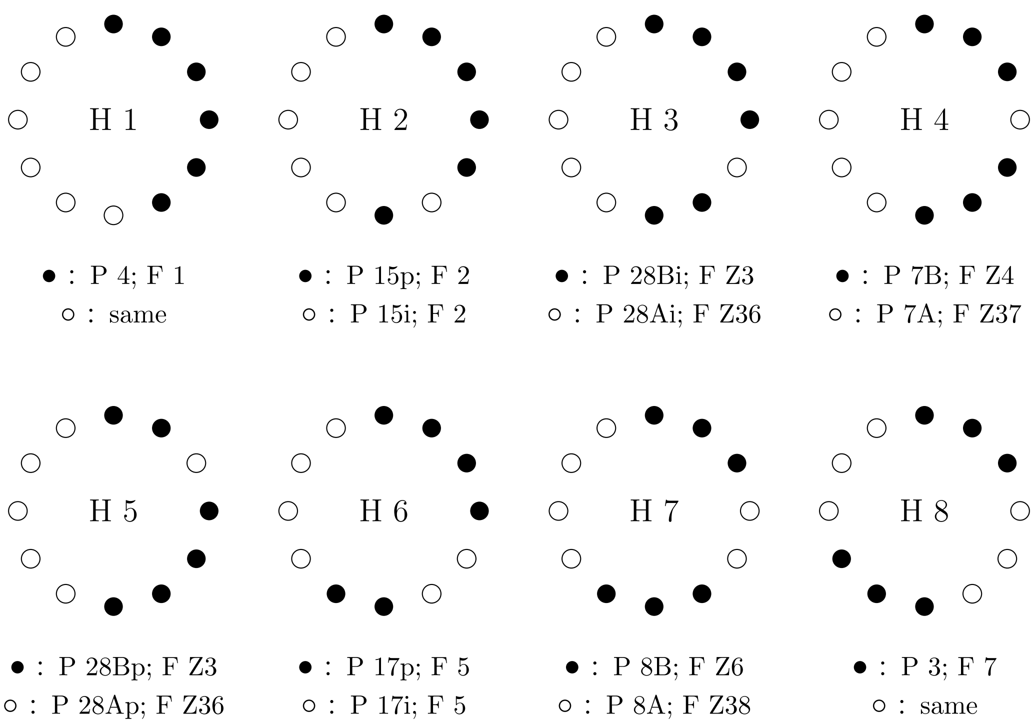


Figure 9: Hauer Tropes 1-8.

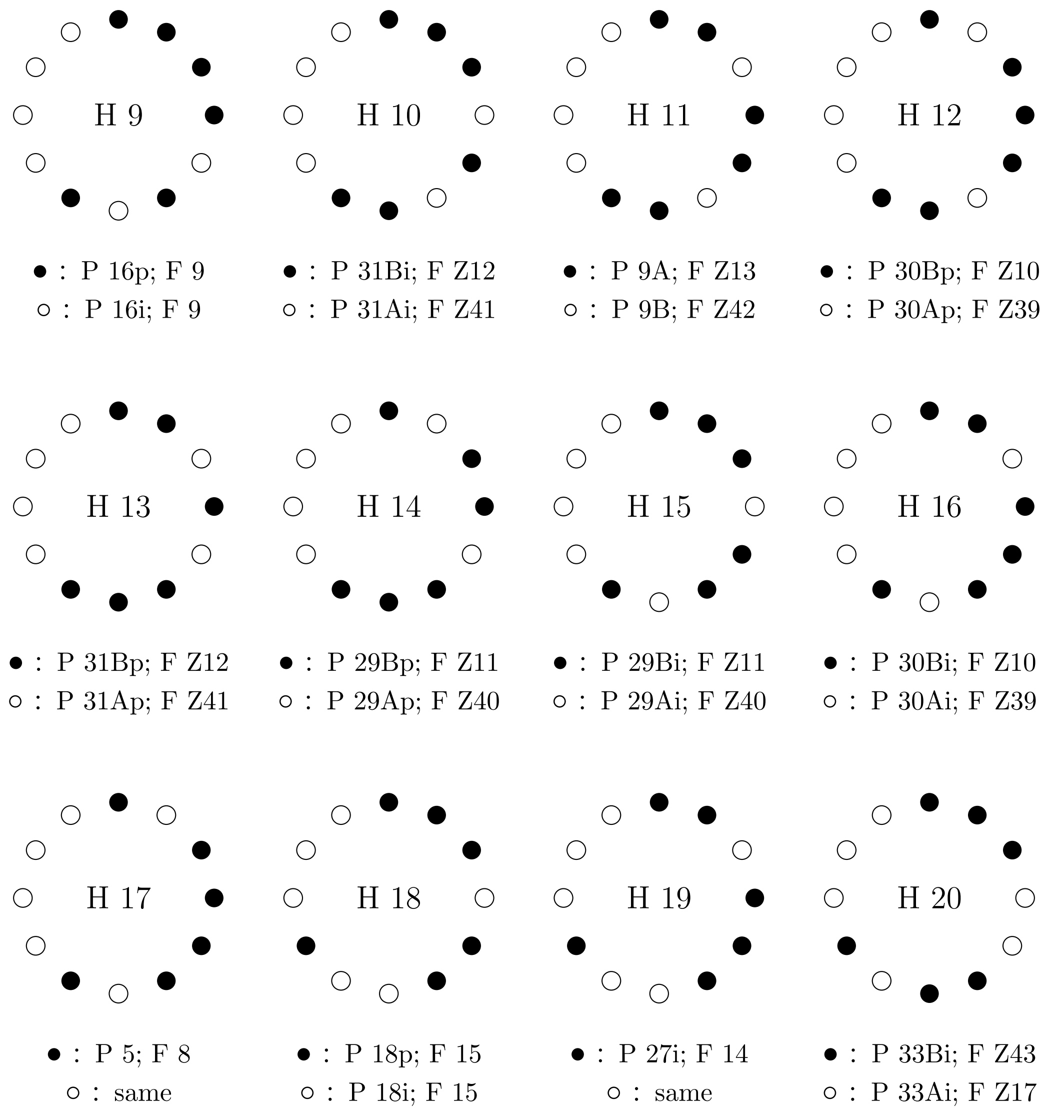


Figure 10: Hauer Tropes 9-20.

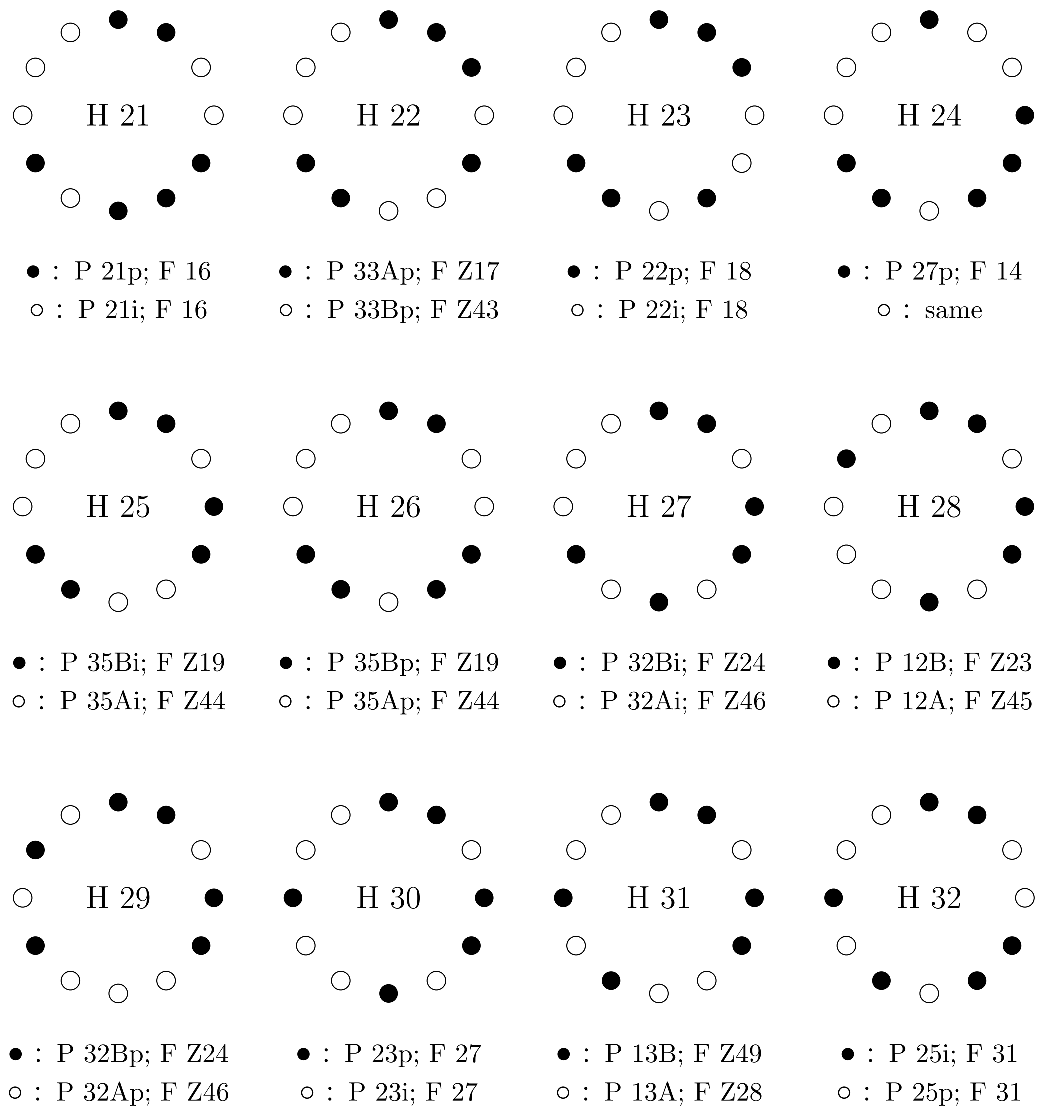


Figure 11: Hauer Tropes 21-32.

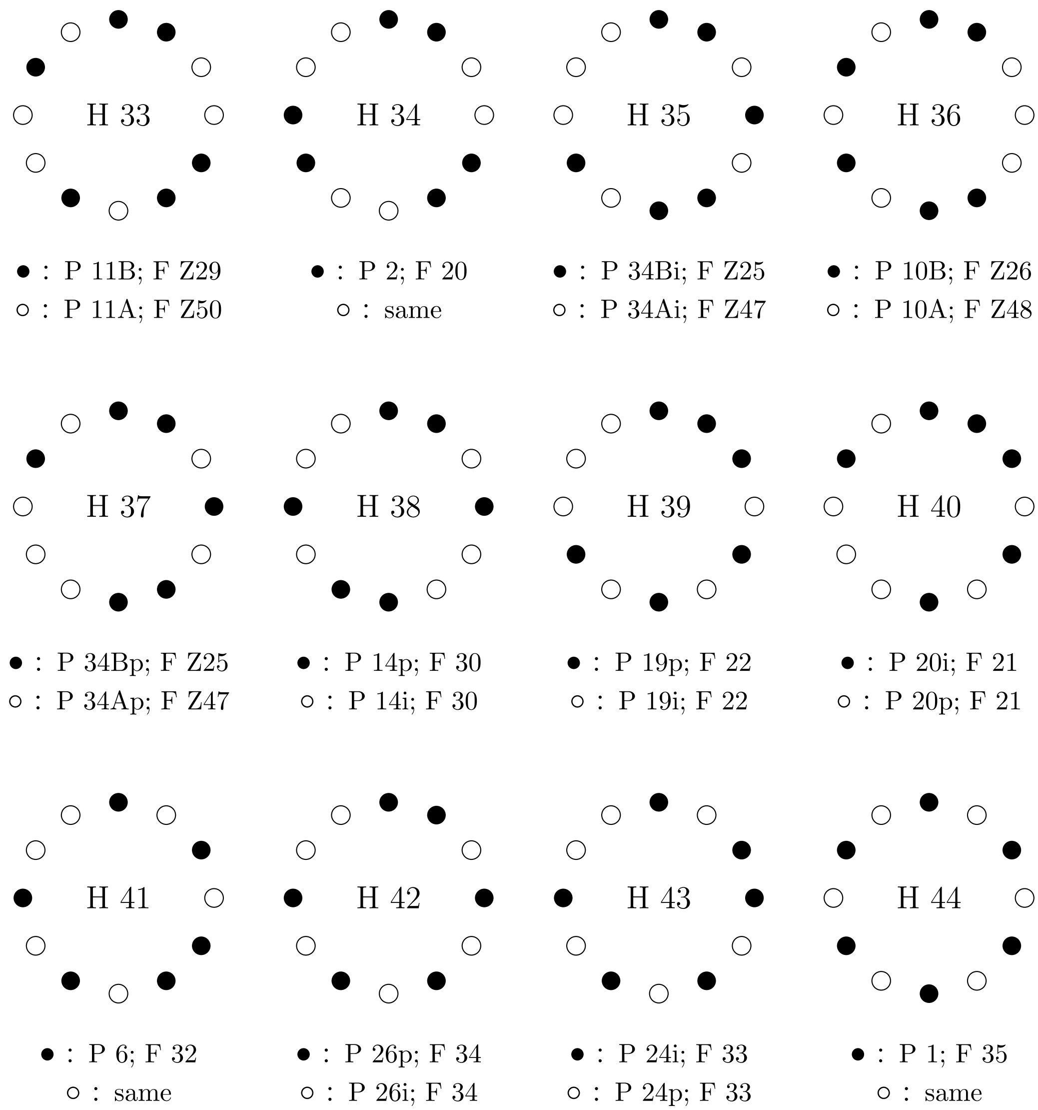


Figure 12: Hauer Tropes 33-44.

B Hexachord Types by Forte Number

			combin.				
Forte	prime form	interval vector	T	I	RI	Perle	Hauer
1	$\{0,1,2,3,4,5\}$	$\langle 5,4,3,2,1,0\rangle$	•	•	•	4	1.I, II
2	$\{0,1,2,3,4,6\}$	$\langle 4,4,3,2,1,1\rangle$		•		15	2.I, II
Z3	$\{0,1,2,3,5,6\}$	$\langle 4,3,3,2,2,1\rangle$				28B	3.I, 5.I
Z4	$\{0,1,2,4,5,6\}$	$\langle 4,3,2,3,2,1\rangle$			•	7B	4.I
5	$\{0,1,2,3,6,7\}$	$\langle 4,2,2,2,3,2\rangle$		•		17	6.I, II
Z6	$\{0,1,2,5,6,7\}$	$\langle 4,2,1,2,4,2\rangle$			•	8B	7.I
7	$\{0,1,2,6,7,8\}$	$\langle 4,2,0,2,4,3\rangle$	•	•	•	3	8.I, II
8	$\{0,2,3,4,5,7\}$	$\langle 3,4,3,2,3,0\rangle$	•	•	•	5	17.I, II
9	$\{0,1,2,3,5,7\}$	$\langle 3,4,2,2,3,1\rangle$		•		16	9.I, II
Z10	$\{0,1,3,4,5,7\}$	$\langle 3,3,3,3,2,1\rangle$				30B	12.I, 16.I
Z11	$\{0,1,2,4,5,7\}$	$\langle 3,3,3,2,3,1\rangle$				29B	14.I, 15.I
Z12	$\{0,1,2,4,6,7\}$	$\langle 3,3,2,2,3,2\rangle$				31B	10.I, 13.I
Z13	$\{0,1,3,4,6,7\}$	$\langle 3,2,4,2,2,2\rangle$			•	9A	11.I
14	$\{0,1,3,4,5,8\}$	$\langle 3,2,3,4,3,0\rangle$	•			27	19.I, II; 24.I, II
15	$\{0,1,2,4,5,8\}$	$\langle 3,2,3,4,2,1\rangle$		•		18	18.I, II
16	$\{0,1,4,5,6,8\}$	$\langle 3,2,2,4,3,1\rangle$		•		21	21.I, II
Z17	$\{0,1,2,4,7,8\}$	$\langle 3,2,2,3,3,2\rangle$				33A	20.II, 22.I
18	$\{0,1,2,5,7,8\}$	$\langle 3,2,2,2,4,2\rangle$		•		22	23.I, II

prime form: see Forte, op. cit.

interval vector: see footnote 6 above.

combin.: combinatoriality by transposition, inversion, retrograde inversion.

Table 1: Hexachord Types by Forte Number (1 of 3).

Forte	prime form	interval vector	combin. T	combin. I	combin. RI	Perle	Hauer
Z19	$\{0,1,3,4,7,8\}$	$\langle 3,1,3,4,3,1\rangle$				35B	25.I, 26.I
20	$\{0,1,4,5,8,9\}$	$\langle 3,0,3,6,3,0\rangle$	•	•	•	2	34.I, II
21	$\{0,2,3,4,6,8\}$	$\langle 2,4,2,4,1,2\rangle$		•		20	40.I, II
22	$\{0,1,2,4,6,8\}$	$\langle 2,4,1,4,2,2\rangle$		•		19	39.I, II
Z23	$\{0,2,3,5,6,8\}$	$\langle 2,3,4,2,2,2\rangle$			•	12B	28.I
Z24	$\{0,1,3,4,6,8\}$	$\langle 2,3,3,3,3,1\rangle$				32B	27.I, 29.I
Z25	$\{0,1,3,5,6,8\}$	$\langle 2,3,3,2,4,1\rangle$				34B	35.I, 37.I
Z26	$\{0,1,3,5,7,8\}$	$\langle 2,3,2,3,4,1\rangle$			•	10B	36.I
27	$\{0,1,3,4,6,9\}$	$\langle 2,2,5,2,2,2\rangle$		•		23	30.I, II
Z28	$\{0,1,3,5,6,9\}$	$\langle 2,2,4,3,2,2\rangle$			•	13A	31.II
Z29	$\{0,2,3,6,7,9\}$	$\langle 2,2,4,2,3,2\rangle$			•	11B	33.I
30	$\{0,1,3,6,7,9\}$	$\langle 2,2,4,2,2,3\rangle$		•		14	38.I, II
31	$\{0,1,4,5,7,9\}$	$\langle 2,2,3,4,3,1\rangle$		•		25	32.I, II
32	$\{0,2,4,5,7,9\}$	$\langle 1,4,3,2,5,0\rangle$	•	•	•	6	41.I, II
33	$\{0,2,3,5,7,9\}$	$\langle 1,4,3,2,4,1\rangle$		•		24	43.I, II
34	$\{0,1,3,5,7,9\}$	$\langle 1,4,2,4,2,2\rangle$		•		26	42.I, II
35	$\{0,2,4,6,8,10\}$	$\langle 0,6,0,6,0,3\rangle$	•	•	•	1	44.I, II
Z36	$\{0,1,2,3,4,7\}$	$\langle 4,3,3,2,2,1\rangle$				28A	3.II, 5.II
Z37	$\{0,1,2,3,4,8\}$	$\langle 4,3,2,3,2,1\rangle$			•	7A	4.II
Z38	$\{0,1,2,3,7,8\}$	$\langle 4,2,1,2,4,2\rangle$			•	8A	7.II
Z39	$\{0,2,3,4,5,8\}$	$\langle 3,3,3,3,2,1\rangle$				30A	12.II, 16.II
Z40	$\{0,1,2,3,5,8\}$	$\langle 3,3,3,2,3,1\rangle$				29A	14.II, 15.II
Z41	$\{0,1,2,3,6,8\}$	$\langle 3,3,2,2,3,2\rangle$				31A	10.II, 13.II
Z42	$\{0,1,2,3,6,9\}$	$\langle 3,2,4,2,2,2\rangle$			•	9B	11.II

prime form: see Forte, op. cit.

interval vector: see footnote 6 above.

combin.: combinatoriality by transposition, inversion, retrograde inversion.

Table 2: Hexachord Types by Forte Number (2 of 3).

			combin.				
Forte	prime form	interval vector	T	I	RI	Perle	Hauer
Z43	$\{0,1,2,5,6,8\}$	$\langle 3,2,2,3,3,2\rangle$				33B	20.I, 22.II
Z44	$\{0,1,2,5,6,9\}$	$\langle 3,1,3,4,3,1\rangle$				35A	25.II, 26.II
Z45	$\{0,2,3,4,6,9\}$	$\langle 2,3,4,2,2,2\rangle$			•	12A	28.II
Z46	$\{0,1,2,4,6,9\}$	$\langle 2,3,3,3,3,1\rangle$				32A	27.II, 29.II
Z47	$\{0,1,2,4,7,9\}$	$\langle 2,3,3,2,4,1\rangle$				34A	35.II, 37.II
Z48	$\{0,1,2,5,7,9\}$	$\langle 2,3,2,3,4,1\rangle$			•	10A	36.II
Z49	$\{0,1,3,4,7,9\}$	$\langle 2,2,4,3,2,2\rangle$			•	13B	31.I
Z50	$\{0,1,4,6,7,9\}$	$\langle 2,2,4,2,3,2\rangle$			•	11A	33.II

prime form: see Forte, op. cit.

interval vector: see footnote 6 above.

combin.: combinatoriality by transposition, inversion, retrograde inversion.

Table 3: Hexachord Types by Forte Number (3 of 3).